# MODELING OF ANISOTROPIC TURBULENT TRANSPORT IN SIMULATIONS OF LIQUID METAL FLOWS IN MAGNETIC FIELDS

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Synopsis: The paper deals with an extended Reynolds stress closure for modeling of turbulent transport in engineering MHD applications. The model was implemented in a commercial CFD code, and predictions were compared with experimental data. Properties and limitations of the model are discussed, especially in view of the extreme anisotropies encountered in MHD turbulence.

Keywords: turbulence model, numerical simulation, turbulent transport, stationary magnetic field

## 1. Introduction

Electrostatic magnetic fields are used in continuous casting of steel to brake and control the mean flow of liquid metal in the mould. The magnetic field also causes magnetic Joule dissipation of turbulence, thus affecting turbulent transport of heat and mass. Numerical simulations of this and other turbulent magnetohydrodynamic (MHD) flows generally suffer from the inability of conventional turbulence models (like the two-equation $K$-$\varepsilon$ model, or full Reynolds stress transport models) to deal with the large anisotropies of length scales encountered in MHD turbulence.

To improve the situation, it has been suggested to include information about length-scale anisotropy in an extended Reynolds stress model for MHD applications. The so-called dimensionality tensor was first introduced by Reynolds and co-workers [1] to help describe the effect of rapid rotation on turbulence. While the well-known Reynolds stress tensor accounts for the kinetic energy of fluctuations in different directions, the dimensionality tensor carries information about the length-scales in different directions. The latter information has proved vital for a correct description of magnetic Joule dissipation of turbulence [2].

A conventional Reynolds stress transport (RST) model includes transport equations for the six independent Reynolds stresses and the scalar viscous dissipation rate. The RST model presented here has been extended with a model transport equation for a dimensionality anisotropy variable, which is related to the field-parallel component of the dimensionality tensor. The new anisotropy variable enables accurate modeling of the Joule dissipation tensor, and may in principle be used to improve also other model terms in the Reynolds stress equations.

Both the full Reynolds stress closure, and a three-equation $K$-$\varepsilon$ version of the model have been implemented in a commercial CFD solver. The model is demonstrated on two idealized flow cases: decaying homogeneous turbulence and plane channel flow. The properties of the model are currently being investigated; we therefore end with a discussion of preliminary conclusions and directions for future work.

## 2. Theory and modeling

Reynolds averaging of the incompressible Navier-Stokes equations yields the Reynolds averaged equations for transport of momentum and a passive scalar $\Phi$,

$$\frac{\mathrm{D}U_i}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\nu S_{ij} - \overline{u_i u_j}\right) + F_i \ , \quad (\text{where } S_{ij} \equiv \frac{1}{2}\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right]) \qquad (1)$$

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} = \frac{\partial}{\partial x_j}\left( \Gamma_\Phi \frac{\partial \Phi}{\partial x_j} - \overline{u_i \phi} \right) + S_\Phi . \tag{2}$$

$F_i$ is here the Lorentz force, and $S_\Phi$ is a volume source of the passive scalar (where applicable). In so-called second-moment or Reynolds stress closures, transport equations are solved for the individual Reynolds stresses $R_{ij} \equiv \overline{u_i u_j}$ (gives turbulent kinetic energy as $K \equiv R_{ii}/2$) and the scalar viscous dissipation rate, $\varepsilon$. The turbulent flux term in the scalar transport equation is usually modeled with a simple expression in the Reynolds stress tensor (Daly-Harlow, [3]):

$$\overline{u_i \phi} = -c_\phi \frac{K^2}{\varepsilon} \frac{\overline{u_i u_k}}{K} \frac{\partial \Phi}{\partial x_k} . \tag{3}$$

This makes the scalar diffusion anisotropic, and it is largely governed by the turbulent velocity fluctuations in a particular direction. The quality of turbulent transport predictions thus depends on the accuracy with which we can predict the Reynolds stresses.

Transport equations for the Reynolds stresses may be written

$$\begin{aligned} \frac{\mathrm{D}R_{ij}}{\mathrm{D}t} = & \underbrace{-R_{ik}\frac{\partial U_j}{\partial x_k} - R_{jk}\frac{\partial U_i}{\partial x_k}}_{P_{ij}} + \underbrace{\left(\overline{f_i u_j} + \overline{f_j u_i}\right)/\rho}_{\Psi_{ij}} + \underbrace{\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}}_{\Pi_{ij}} \\ & \underbrace{-\frac{\partial}{\partial x_k}\left(\overline{u_i u_j u_k} + \frac{\overline{p u_j}}{\rho}\delta_{ik} + \frac{\overline{p u_i}}{\rho}\delta_{jk}\right)}_{d_{ij}} + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial R_{ij}}{\partial x_k}\right) - \underbrace{2\nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}}}_{\varepsilon_{ij}} \end{aligned} \tag{4}$$

The pressure-strain rate term $\Pi_{ij}$ is traceless. It gives no net contribution to the turbulent kinetic energy, but it acts to redistribute energy among the Reynolds stress components. A Poisson equation for fluctuating pressure can be used to eliminate $p$ to obtain an exact expression for $\Pi_{ij}$. The latter can be split into three parts,

$$\Pi_{ij} = \Pi_{ij}^{(s)} + \Pi_{ij}^{(r)} + \Pi_{ij}^{(m)}, \tag{5}$$

where $\Pi_{ij}^{(s)}$ (slow pressure-strain) represents nonlinear turbulence-turbulence interaction, $\Pi_{ij}^{(r)}$ (rapid pressure-strain) interaction with the mean flow, and $\Pi_{ij}^{(m)}$ interaction with fluctuating Lorentz forces. The first two terms are the same as in ordinary hydrodynamic turbulence.

In conventional turbulence modeling, a lot of attention is given the rapid pressure-strain,

$$\Pi_{ij}^{(r)} = -\frac{1}{2\pi}\int \frac{\partial U_k'}{\partial x_l'} \overline{\frac{\partial u_l'}{\partial x_k'}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \frac{dV'}{r} \approx \frac{\partial U_k}{\partial x_l}\left(M_{ilkj} + M_{jlki}\right) \tag{6}$$

The latter part of Eq. 6 is the homogeneous approximation, introducing the fourth-rank M-tensor, defined in physical and Fourier space as

$$M_{ijpq} \equiv -\frac{1}{4\pi}\int \frac{\partial^2 \overline{u_i u_j'}}{\partial r_p \partial r_q}\frac{dV'}{r} = \int \frac{k_p k_q}{k^2}\hat{E}_{ij} d^3k , \tag{7}$$

where $\hat{E}_{ij} \equiv \overline{\hat{u}_i^* \hat{u}_j}$ is the spectral energy tensor. The challenge is here to model the M-tensor in terms of primary variables included in the closure model. A conventional Reynolds stress transport (RST) model includes transport equations for the six independent components of the Reynolds stress tensor, and for the scalar viscous dissipation, $\varepsilon \equiv \varepsilon_{ii}/2$. A model of **M** will then have to be a tensorial function in terms of $R_{ij}$, with function coefficients dependent on scalars like $\varepsilon$ and the turbulent kinetic energy, $K \equiv R_{ii}/2$. In the following, we use the simplest "standard" RST closure, with model terms and dissipation-rate transport equation defined in Table 1 (the so-called IP model for rapid pressure-strain, and the linear Rotta model for slow pressure-strain).

$$\varepsilon_{ij} = \frac{2}{3}\delta_{ij} \qquad d_{ij} = \frac{\partial}{\partial x_k}\left( c_s \frac{K}{\varepsilon} R_{kl} \frac{\partial R_{ij}}{\partial x_l} \right) \qquad \Pi_{ij}^{(s)} = -c_1 \varepsilon \left( \frac{R_{ij}}{K} - \frac{2}{3}\delta_{ij} \right) \qquad \Pi_{ij}^{(r)} = -0.6\left( P_{ij} - \frac{2}{3}P_{kk} \right)$$

$$\frac{D\varepsilon}{Dt} = c_\varepsilon \frac{\varepsilon}{K} P_{kk} - c_{\varepsilon 2}\frac{\varepsilon^2}{K} + \frac{\partial}{\partial x_k}\left( c_\varepsilon \frac{K}{\varepsilon} \overline{u_k u_l} \frac{\partial \varepsilon}{\partial x_l} \right)$$

***Table 1:*** *"Standard" RST closure (see Launder, Reece & Rodi [4]).*

In the MHD case, it is practical to treat the magnetic pressure-strain contribution together with the force term $\Psi_{ij}$. This gives us the Joule dissipation tensor,

$$\mu_{ij} = -\Psi_{ij} - \Pi_{ij}^{(m)}. \tag{8}$$

The Joule dissipation can also be expressed in terms of the M-tensor; spectral analysis give

$$\mu_{ij} = 2\frac{\sigma}{\rho}\int \frac{(B_n k_n)^2}{k^2}\hat{E}_{ij} d^3k = 2\frac{\sigma B^2}{\rho} n_p n_q M_{ijpq}, \qquad n_i = \frac{B_i}{B}, \tag{9}$$

with electric conductivity $\sigma$ and magnetic flux density $B_i$. The scalar Joule dissipation of turbulent kinetic energy is

$$\mu \equiv \frac{\mu_{ii}}{2} = \frac{\sigma B^2}{\rho} n_p n_q M_{iipq} = \frac{\sigma B^2}{\rho} n_p n_q Y_{pq}. \tag{10}$$

We here introduce the *Reynolds dimensionality tensor*, $Y_{ij} \equiv M_{nnij}$, as a contraction of **M** in the first index pair [1]. Note that contraction of the last index pair yields the Reynolds stress tensor, $R_{ij} = M_{ijnn}$. The two tensors **Y** and **R** are tensorially independent. While **R** accounts for the *magnitude* of velocity fluctuations in different directions, the "componentality" of turbulence, **Y** carries information about the distribution of *length scales* in different directions, the "dimensionality" of turbulence [1]. From Eq. 10 we conclude *(i)* that the scalar Joule dissipation can be expressed exactly in terms of one single component of **Y**, namely $n_p n_q Y_{pq}$, and *(ii)* that the information contained in **R** alone does not suffice to model MHD turbulence.

In addition to equations for $R_{ij}$ and $\varepsilon$, the MHD turbulence model we propose includes a transport equation for a scalar dimensionality anisotropy variable defined as

$$\alpha \equiv n_i n_k Y_{ki}/(2K) \quad \text{(so that } \mu \equiv 2\sigma B^2 K\alpha/\rho\text{)}. \tag{11}$$

The Joule dissipation tensor in the RST equations is modeled using a tensor function in $R_{ij}$, $\alpha$, and the unit direction vector $n_i$; see Table 2. The standard model equation for $\varepsilon$ (Table 1) is complemented with a destruction term (last), due to Joule dissipation (with $c_{\varepsilon\alpha} = 0.5$):

$$\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_k}\left( c_\varepsilon \frac{K}{\varepsilon}\overline{u_k u_l}\frac{\partial \varepsilon}{\partial x_l} \right) + c_\varepsilon \frac{\varepsilon}{K}P_{kk} - c_{\varepsilon 2}\frac{\varepsilon^2}{K} - c_{\varepsilon\alpha}\frac{\varepsilon}{K}\frac{\sigma B^2}{\rho}\alpha K. \tag{12}$$

An exact transport equation for $\alpha$ can be derived from the Navier-Stokes equations, but all terms require modeling. The model equation is of the generic form (2), with source terms due to rapid pressure-strain, production, Joule dissipation, and return-to-isotropy; see Table 3.

Joule dissipation tensor:

$$\mu_{ij} = 2\frac{\sigma B^2}{\rho}\left(GR_{ij} + \frac{H}{2}\left[n_i n_k R_{kj} + n_j n_k R_{ki}\right]\right)$$

Scalar Joule dissipation:

$$\mu \equiv 2\frac{\sigma B^2}{\rho}\alpha K$$

$$\begin{cases} G(\alpha, I_{na}) = \alpha + \dfrac{27}{20}\alpha^2(1-\alpha)\left(I_{na} + \dfrac{2}{3}\right) \\ H(\alpha) = -\dfrac{27}{10}\alpha^2(1-\alpha) \\ I_{na} = \dfrac{n_i n_k R_{ki}}{K} - \dfrac{2}{3} \end{cases}$$

***Table 2:*** *Model expression for the Joule dissipation tensor [2].*

$$S_\alpha = P_\alpha + \Pi_\alpha^{(r)} + \pi_\alpha + \mu_\alpha \qquad \left(\text{Mean strain-rate:} \quad S_{ij} \equiv \frac{1}{2}\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right]\right)$$

Return-to-isotropy: ($c_{\alpha 2} = 0.2$)

$$\pi_\alpha = c_{\alpha 2}\frac{\varepsilon}{K}\left(\frac{1}{3} - \alpha\right)$$

Production:

$$P_\alpha = -2\frac{G}{K}R_{ij}S_{ji} - 2\frac{H}{K}n_i n_j R_{jk}S_{ki} + \frac{\alpha}{K}R_{ij}S_{ji}$$

Joule destruction: ($c_{\alpha 1} = 0.87$)

$$\mu_\alpha = 2c_{\alpha 1}\frac{\sigma B^2}{\rho}\alpha^2$$

Rapid pressure-strain:

$$\Pi_\alpha^{(r)} = 2\alpha\left(-\frac{3}{2}\alpha^2 + \frac{29}{10}\alpha - \frac{7}{5}\right)n_i n_j S_{ji}$$

***Table 3:*** *Source terms for the $\alpha$ transport equation (functions G and H defined in Table 2).*

## 3. Model predictions

The model has been compared with direct numerical simulations (DNS) of decaying homogeneous turbulence [2] (DNS data by Schumann, [5]). Fig. 1a shows the decay of turbulent kinetic energy for different initial interaction numbers, $N \equiv \sigma B^2 K/\rho\varepsilon$. Fig. 1b shows the ratio between Reynolds stresses parallel (n) and perpendicular (p) to the direction of the magnetic field.

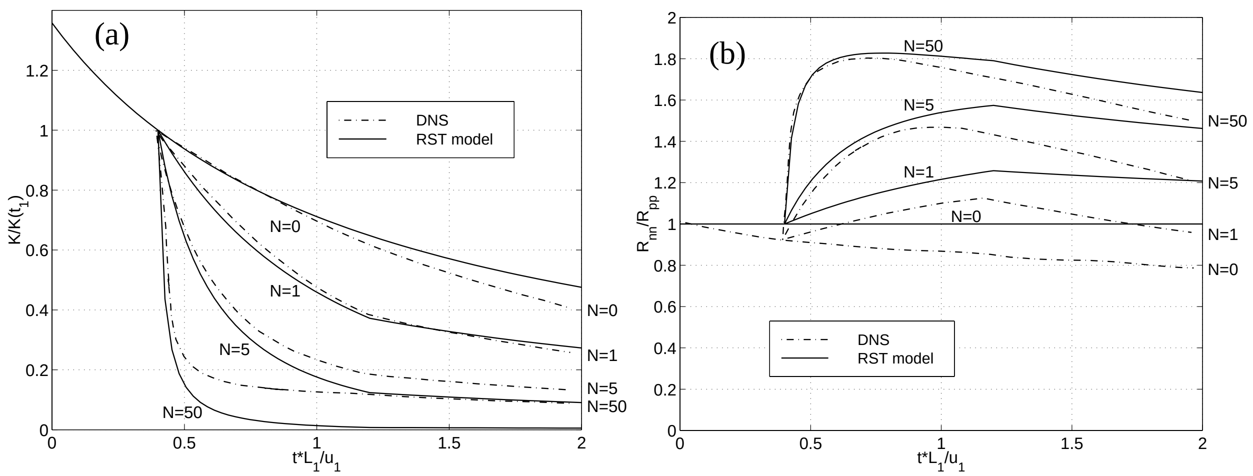


***Fig. 1:*** *Decay of homogeneous turbulence. Magnetic field is on between times 0.4 and 1.2. (a) turbulent kinetic energy; (b) ratio of Reynolds stresses in parallel and perpendicular directions.*

The model has recently been implemented in a commercial CFD code, *CFX 4.2*, together with equations for mean electrostatic potential and mean Lorentz force. To illustrate some properties of the model, Figs. 2a-d show results of a 2D simulation of a 4 cm high channel in a transverse magnetic field; the magnetic field is perpendicular to the plane of flow (channel assumed very wide). Length of magnet is 30 cm, fluid is mercury. $\mathrm{Re} = UL/\nu = 2\cdot 10^5$, $N = \sigma B^2 L/\rho U = 2.45$, and $\mathrm{Ha} = \sqrt{N\,\mathrm{Re}} = 700$. The simulation uses standard wall functions ($y^+ \approx 25-40$); in analogy with other turbulence equations, we use zero flux wall boundary conditions for the $\alpha$ equation.

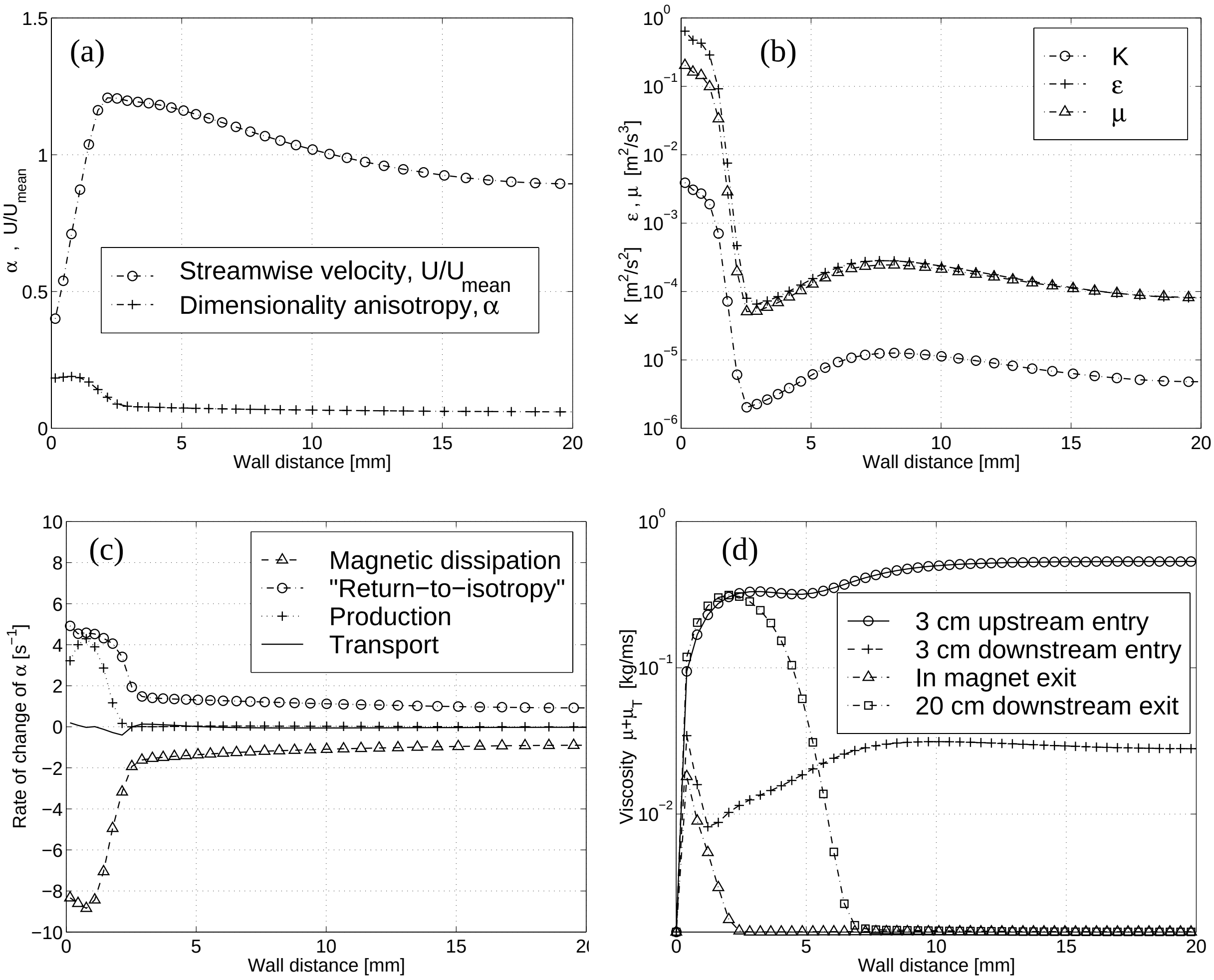


***Fig. 2:*** *Variable profiles in the middle of the magnet are shown in (a) and (b); (c) shows budget of the $\alpha$ equation in the middle of the magnet ($\Pi_\alpha^{(r)} = 0$ in this 2D example). The sum of molecular and turbulent viscosity is shown in (d) for a few positions relative to the magnet. The baseline of (d) represents molecular viscosity; the core flow appears to be laminarized in the magnet, and turbulence is restored by production downstream of the magnet.*

## 4. Discussion

This paper describes the current state of on-going work. We are currently exploring the properties of the new model, and we will here limit ourselves to some general and preliminary observations.

In the model equation for $\alpha$, the two strain-related terms $P_\alpha$ and $\Pi_\alpha^{(r)}$ are modeled using the assumption that the full **Y** tensor is axisymmetric about the magnetic field direction, $n_i$. This assumption is reasonable when Joule dissipation is the dominating mechanism for structural deformation. In that case, however, their contribution to the source term should be small, so that it may not be worth the extra computational effort to include them in the modeling.

As noted above, the prediction of anisotropic diffusion with the Daly-Harlow hypothesis (3) will never be more accurate than the prediction of the individual Reynolds stresses. In the current treatment of the pressure-strain term (5) in the Reynolds stress equations, we neglect the influence of walls. There is reflection of fluctuating pressure near the wall, so that the wall-normal component of the Reynolds stresses will be damped (or rather the energy transferred into parallel components). This process is probably more important in MHD, than in ordinary hydrodynamic turbulence, as the Joule dissipation drives the turbulence to a virtually two-dimensional (2D) state. In a channel flow, for example, structures may eventually grow as large as the channel width. One may argue that damping of the wall-normal velocity fluctuations will then force the turbulence close to a two-component (2C) state, throughout the width of the channel. Various channel experiments also suggest that fluctuations may be almost completely killed in the magnetic field direction, while fluctuations in the perpendicular plane persist for long times. Note that this behavior is radically different from the homogeneous case without walls, where the turbulence tends to become two-dimensional, but remains three-component (2D-3C). With some effort, it should be possible to devise a model term for the damping of the wall-normal velocity fluctuations. To be successful, however, this also requires more complex models for other terms in the Reynolds stress equations; of the terms in Table 1, neither the two pressure-strain terms, nor the viscous dissipation term, can deal with large anisotropies. They all produce unphysical results in the two-component limit. A more accurate model for the rapid pressure-strain should preferably be sensitive also to the dimensional state, for example by including $\alpha$ or other dimensional information in the modeling.

Considering the complications above, an extended eddy viscosity $K$–$\varepsilon$ model could be an attractive alternative to a complex and computationally expensive RST model. One would then use the transport equation for $\alpha$, together with the already presented magnetic source terms in the $K$ and $\varepsilon$ equations. A phenomenological model for anisotropic scalar diffusion could be devised, for example, by exchanging the factor $\overline{u_i u_j}/K$ in the Daly-Harlow expression (3) with a model function $d_{ij}$, say. To reflect the effect of the magnetic field on anisotropy, $d_{ij}$ should be axisymmetric about $n_i$. The model coefficients may be functions of $K$, $\varepsilon$, $\alpha$, local interaction number $N \equiv \sigma B^2 K/\rho\varepsilon$, etc., and possibly some relation between a turbulence length scale and wall distance. Similar modifications may be motivated also for the Boussinesq hypothesis (relating Reynolds stress anisotropy to the mean strain rate).

The model results so far are encouraging. Future work will focus on validation with experimental data and direct numerical simulations, as available.

## 5. References


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